MULTI-FOLD SUMS FROM A SET WITH FEW PRODUCTS

LIANGPAN LI

Abstract. In this paper we show that for any $k \geq 2$, there exist two universal constants $C_k, D_k > 0$, such that for any finite subset A of positive real numbers with $|AA| \leq M|A|$, $|kA| \geq \frac{C_k}{M^{D_k}} \cdot |A|^{\log_4 2k}$.

1. Introduction

We begin with some notation: Given a finite subset A of some commutative ring, we let $A \star A$ denote the set $\{a \star b : a, b \in A\}$, where $\star$ is a binary operation on A. When three or more summands or multiplicands are used, we let kA denote the k-fold sum-set $A + A + \cdots + A$, and let $A^{(k)}$ denote the k-fold product-set $AA \cdots A$.

Erdős and Szemerédi ([8]) once conjectured that for any $\alpha < 2$, there exists a universal constant $C_\alpha > 0$, such that for finite subset A of real numbers,

$$\max\{|A + A|,\ |AA|\} \geq C_\alpha |A|^\alpha.$$

Non-trivial lower bounds for α were achieved by many authors such as by Erdős and Szemerédi ([8], qualitatively), Nathanson ([14], 32/31), Ford ([9], 16/15), Chen ([3], 6/5), Elekes ([5], 5/4), and Solymosi ([17], $14/11 - o(1)$; [18], $4/3 - o(1)$).

Another type of question than one can attack regarding sums and products is to either assume that the sum-set $A + A$ is very small, and then to show that the product-set AA is very large, or to suppose that AA is very small, and then to show that $A + A$ is very large. The best two results toward this question are respectively due to Elekes and Ruzsa ([7]), who fully confirmed the first part of the question, and by Chang ([2]), who solved the second part of the question in the setting of integers.

Similarly, one can consider multi-fold sums and products, but very few results are known especially in the setting of reals. Let B be a finite subset of integers, then Chang ([2]) showed that if $|BB| \leq |B|^{1+\epsilon}$, then the multi-fold sum-set $|kB| \gg_{\epsilon,k} |B|^{n-\delta}$, where $\delta \to 0$ as $\epsilon \to 0$; and Bourgain and Chang ([1]) proved that for any $b \geq 1$, there exists $k \in \mathbb{N}$ independent of B such that $|kB| \cdot |B^{(k)}| \geq |B|^b$. At the moment how to extent these results to the real numbers is not known yet. Recently, Croot and Hart established in [4] the following interested result:

2010 *Mathematics Subject Classification*: 11B75.

Key words and phrases: Erdős-Szemerédi conjecture, sum-product estimate, sum-set, product-set.

Theorem 1.1. *For all $k \geq 2$ and $\epsilon \in (0, \epsilon_0(k))$ we have that the following property holds for all $n > n_0(k, \epsilon)$: If A is a set of n real numbers and $|AA| \leq n^{1+\epsilon}$, then*

$$|kA| \geq n^{\log_4 k - f_k(\epsilon)},$$

where $f_k(\epsilon) \to 0$ as $\epsilon \to 0$.

Croot and Hart also remarked that they have several different approaches to proving a theorem of the quality of Theorem 1.1.

The purpose of the present paper is to give the following slight improvement of the above Croot-Hart theorem in a rather elementary way. Our idea comes from Solymosi's wonderful proof ([18]) of the best currently known sum-product estimates of real numbers mentioned earlier. Solymosi's idea has appeared elsewhere in [11] and [13].

Theorem 1.2. *For any $k \geq 2$, there exist three positive universal constants C_k, D_k, Ψ_k, such that for any finite subset A of positive real numbers with $|AA| \leq M|A|$,*

$$|kA| \geq \frac{C_k}{M^{D_k}} \cdot |A|^{\Psi_k}.$$

With $\Psi_1 \triangleq 1$, the constants $\{\Psi_k\}_{k \geq 2}$ can be generated in any of the following way:

$$\Psi_k = \frac{1 + \Psi_{k_1} + \Psi_{k_2}}{2} \quad (k_1 + k_2 = k).$$

Particularly, we can take $\Psi_k = \log_4 2k$.

There are some other interested estimates on sum-sets and product-sets in the reals. For example, see [6], [12], [15] and [16].

2. Proof of the main theorem

We will prove Theorem 1.2 for all $k \in \mathbb{N}$ by induction. Obviously, one can choose $D_1 = 0$, $C_1 = \Psi_1 = 1$. Next for any $k \geq 2$, we assume the existences of positive universal constants C_i, D_i and Ψ_i for all $i \in [2, k)$. Our purpose is to find C_k, D_k and Ψ_k satisfying the required property. Let k_1, k_2 be any two natural numbers such that $k_1 + k_2 = k$.

By the Ruzsa triangle inequality, $|A/A| \leq M^2|A|$. For any $s \in A/A$, let $A_s \triangleq \{(x, y) \in A \times A : y = sx\}$. Let $D = \{s : |A_s| \geq \frac{|A|}{2M^2}\}$, and let $s_1 < s_2 < \cdots < s_m$ denote the elements of D, labeled in increasing order. Obviously,

$$\sum_{s \in D} |A_s| \geq \frac{|A|^2}{2},$$

which implies $m \geq \frac{|A|}{2}$. Let A_{m+1} be the projection of A_m onto the vertical line $x = \min A$, and let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection map from $\mathbb{R}^2$ onto the vertical axis. It is geometrically evident that $\{k_1 A_j + k_2 A_{j+1}\}_{j=1}^m$ are mutually disjoint. Thus

$$|(kA) \times (kA)| \geq \sum_{j=1}^m |k_1 A_j + k_2 A_{j+1}| = \sum_{j=1}^m |k_1 A_j| \cdot |k_2 A_{j+1}| = \sum_{j=1}^m |k_1 \Pi(A_j)| \cdot |k_2 \Pi(A_{j+1})|.$$

Note

$$|\Pi(A_j)\Pi(A_j)| \le |AA| \le M|A| \le 2M^3|\Pi(A_j)|.$$

Applying induction to all of the $\Pi(A_j)$'s,

$$|kA|^2 \ge \frac{|A|}{2} \cdot \frac{C_{k_1}}{(2M^3)^{D_{k_1}}}(\frac{|A|}{2M^2})^{\Psi_{k_1}} \cdot \frac{C_{k_2}}{(2M^3)^{D_{k_2}}}(\frac{|A|}{2M^2})^{\Psi_{k_2}},$$

which yields

$$|kA| \ge \Big(\frac{C_{k_1} \cdot C_{k_2}}{2 \cdot (2M^3)^{D_{k_1}+D_{k_2}} \cdot (2M^2)^{\Psi_{k_1}+\Psi_{k_2}}}\Big)^{1/2} \cdot |A|^{\frac{1+\Psi_{k_1}+\Psi_{k_2}}{2}}.$$

Thus one can let $\Psi_k \triangleq \frac{1+\Psi_{k_1}+\Psi_{k_2}}{2}$ and define C_k, D_k in a similar way.

Finally, let $z \triangleq \lfloor \log_2 k \rfloor$. Then

$$\Psi_{2^z} \ge \frac{1}{2} + \Psi_{2^{z-1}} \ge \cdots \ge \frac{z}{2} + \Psi_1 = \frac{z+2}{2} \ge \log_4 2k.$$

Consequently,

$$|kA| \ge |2^z A| \ge \frac{C_{2^z}}{M^{D_{2^z}}} \cdot |A|^{\Psi_{2^z}} \ge \frac{C_{2^z}}{M^{D_{2^z}}} \cdot |A|^{\log_4 2k}.$$

This concludes the whole proof.

Acknowledgements. The author was supported by the NSF of China (11001174).

References

[1] J. Bourgain, M.-C. Chang, *On the size of k-fold sum and product sets of integers*, J. Amer. Math. Soc. 17 (2003) 473–497.

[2] M.-C. Chang, *The Erdős-Szemerédi problem on sum set and product set*, Ann. Math. 157 (2003) 939–957.

[3] Y. G. Chen, *On sums and products of integers*, Proc. Amer. Math. Soc. 127 (1999) 1927–1933.

[4] E. Croot, D. Hart, *h-fold sums from a set with few products*, SIAM J. Discrete Math. 24 (2010) 505–519.

[5] Gy. Elekes, *On the number of sums and products*, Acta Arith. 81 (1997) 365–367.

[6] Gy. Elekes, M. B. Nathanson, I. Z. Ruzsa, *Convexity and sumsets*, J. Number Theory 83 (1999) 194–201.

[7] Gy. Elekes, I. Z. Ruzsa, *Few sums, many products*. Studia Sci. Math. Hungar. 40 (2003) 301–308.

[8] P. Erdös and E. Szemerédi, *On sums and products of integers*. In: Studies in Pure Mathematics (Birkhauser, Basel, 1983) 213–218.

[9] K. Ford, *Sums and products from a finite set of real numbers*, Ramanujan J. 2 (1998) 59–66.

[10] M. Z. Garaev, C.-Y. Shen, *On the size of the set* $A(A+1)$, Math. Z. 265 (2010) 125–132.

[11] D. Hart, A. Niziolek, *Some results on the size of sum and product sets of finite sets of real numbers*, Involve 2 (2009) 603–609.

[12] A. Iosevich, O. Roche-Newton, M. Rudnev, *On an application of Guth-Katz theorem*, arXiv:1103.1354, accepted by Math. Research Letters, 2011.

[13] L. Li, J. Shen, *A sum-division estimate of reals*, Proc. Amer. Math. Soc. 138 (2010) 101–104.

[14] M. B. Nathanson, *On sums and products of integers*, Proc. Amer. Math. Soc. 125 (1997) 9–16.

[15] T. Schoen, I. D. Shkredov, *On sumsets of convex sets*, arXiv:1105.3542, 2011.

[16] C.-Y. Shen, *Algebraic methods in sum-product phenomena*, arXiv:0911.2627, to appear in Israel J. Math., 2009.

[17] J. Solymosi, *On the number of sums and products*, Bull. London Math. Soc. 37 (2005) 491–494.

[18] J. Solymosi, *Bounding multiplicative energy by the sumset*, Adv. Math. 222 (2009) 402–408.

Department of Mathematical Sciences, Loughborough University, LE11 3TU, UK
E-mail address: liliangpan@gmail.com